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A New Lifetime Mixture Model to Estimate Repair times using Bayesian Approach

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ABSTRACT

In this study 3-component mixture model using Ailamujia distribution is analyzed under Bayesian paradigm. Joint posteriors are obtained for Jeffreys' and gamma priors. Bayes estimators of mixture parameters and associated Bayes risks are derived using three loss functions i.e squared error loss function (SELF), precautionary loss function (PLF) and DeGroot loss function (DLF). The prior predictive method is used for hyper parameter elicitation . To numerically check the performance of Bayes estimators, simulated results are obtained for different test termination times, parametric values and sample sizes. Two data sets, on repair times of refrigerator components and recovery times of cancer patients are analyzed to numerically exhibit the applicability of proposed mixture model. Results suggest that DLF is a better option for estimating the component parameters.

Keywords: *Mixture Model ; Bayes risk; Loss function; Hyperparameters ; Prior predictive method; Jeffreys' prior*

1. Introduction

A new lifetime probability model by Lv et al. (2002) known as an Ailamujia distribution is an emergent candidate in supportability data analysis in the field of engineering, medical science and quality control. Ailamujia distribution has proven suitable to be applied in some practical situations, such as to model the repair, guarantee and the distribution delay times. For example, Yu et al. (2008) used the Ailamujia distribution to analyze the degree of injury in the battle ground and developed a new method to address the issues related to production and distribution of such injury in campaign macrocosm. As a competitive model, this new distribution has attracted attention of many authors. Long (2015) presented the Bayesian analysis of Ailamujia distribution by taking type-II censoring under three different priors. Rashid et al. (2018) developed a new compound lifetime distribution called Ailamujia Power Series Distribution (APSD). Jayakumar and Elangovan (2019) introduced Area Biased weighted Ailamujia distribution (ABWAD). Rather et al. (2022) studied Exponentiated Ailamujia distribution and explored its properties. Gomma et al. (2023) introduced an alpha power Ailamujia distribution and showed that it is more suitable then existing competing models. Aijaz et al. (2022) analyzed count data by formulating poisson area biased Ailamujia distribution.

The probability density function (p,d,f) and cumulative distribution function (C.D.F) of Ailamujia distribution are given as respectively:

$$f(\mathrm{x};\theta) = 4x\theta^2 \exp(-2\theta x) \qquad \mathrm{x} \geq 0, \theta_i \geq 0 \qquad (1)$$

$$F(\mathrm{x}) = 1 - (1 + 2\theta x)\exp(-2\theta x) \qquad (2)$$

Where θ is an unknown parameter.

Mixture models formally are composed of two or more probability distributions to capture heterogeneity present in the data. Mixture models have widespread applications almost in all fields of life, from economics to medicine, engineering, social sciences and psychology. For example, in genetics, on a chromosome the location of the quantitative traits and interpretation about microarrays both are connected to mixtures. If some specific mechanism is defined on the basis of which observations are allocated to one of the member of population is usually termed as direct application of mixture models. For example, financial returns act differently in crisis and normal situation. When mixtures are only defined for simplicity or mathematical flexibility and we do not assume any underlying mechanism then it refers to an indirect application of mixed models. Finite mixture model is used as flexible model when distributions are heavy tailed, data is heterogeneous and heterogeneity is observed in cluster analysis. Several statisticians have used mixture distribution to analyze different statistical problems. For example, Kanji (1985) used mixture distribution to describe the wind shear data. Noor et.al. (2020) applied Inverted Kumaraswamy (IKum) mixture model to estimate burning velocity of different chemicals. Feroze and Aslam (2020) considered the reliability estimation for the Topp Leone mixture model using Bayesian technique. For analysis and applications of mixture models one can see Castet and Saleh (2010) and references therein.

Censoring is a condition in lifetime data when only partial information is observed. In this context Romeu (2004) and Gijbels (2010) have provided valuable amount of information. Noor and Aslam (2013) analyzed inverse Weibull mixture model using type-I censoring scheme under Bayesian perspective. Noor et.al. (2021) formed a 4-component mixture model to estimate the average number of incidences and deaths for both genders considering different types of cancer diagnosed in Pakistan . Tahir et al. (2019) used doubly censored data to analyze the mixture of Burr-XII distributions under Bayesian setup.

Vast usefulness of mixture modeling motivated us to propose a new versatile 3-component Ailamujia mixture model. The proposed model is thus analyzed under Bayesian setup using censored data.

The main scheme in the article is as follows: A model to be analyzed along with its likelihood function for type-I censoring scheme is given in section 2. The joint posteriors assuming non-informative and informative priors are derived in the same section. Further, section 2 also contains the derived Bayes estimators under three different loss functions. Elicitation of hyperparameters is also part of this section. Simulated and real data results and discussion on results is provided in section 3. Conclusion derived from the study is given in section 4.

2. Materials & Methods

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This section contains the model, likelihood function and estimation of parameters under different loss functions using informative and non-informative priors.

2.1 The model and likelihood function

A finite 3-CAMM for a random variable X is defined as:

$$f(\mathrm{x},\Psi)=\sum_{d=1}^{3} p_d\, f_d(x)\,,$$

$$f(\mathrm{x};\Psi)=\sum_{d=1}^{3} p_d\, 4x\theta_d^2 \exp(-2\theta_d x) \qquad (3)$$

And C.D.F of 3-CAMM is given as:

$$F(\mathrm{x},\Psi)=1-\left(\sum_{d=1}^{3} p_d (1+2\theta_d x)\exp(-2\theta_d x)\right) \qquad (4)$$

Where $\Psi=(\theta_d,\mathrm{p}_d)$, $d=1,2,3$, $p_1,p_2\geq 0$, $p_1+p_2\leq 1.$, $x\geq 0$, $\theta_d\geq 0$

Figure:1 shows the p.d.f and C.D.F of 3-component Ailamujia mixture distribution and it can be viewed that graph of the distribution is positively skewed.

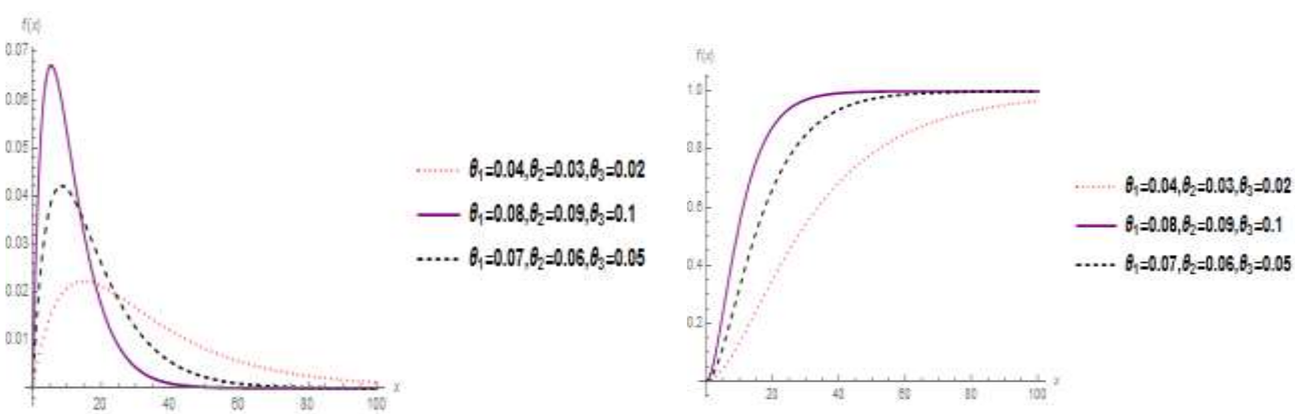

Fig :1 Graph of the p.d.f and C.D.F of 3-component mixture of Ailamujia distribution

Let we assume a life time experiment for a 3-CAMM with a fixed termination time. It is observed that r out of n total units are failed by the end of the experiment and remaining $n-r$ units are functional. So, after inspecting r units (failed items), it is possible to identify

r_1, r_2 and r_3 units and assign them to three subpopulations respectively. Here obviously $r = r_1 + r_2 + r_3$ represent total number of uncensored observations and the remaining $n-r$ observations are known as censored observations. Let define x_{hj} , $0 < x_{hj} \le t$, as the lasting time of the j^{th} $(j = 1, 2, ..., r_h)$ unit that belongs to the h^{th} $(h = 1, 2, 3)$ subpopulation. The likelihood function of a 3-component mixture model for type-I right censored data (Everitt and Hand (1981)) can be written as:

$$L(\Psi \mid \mathrm{x}) \propto \left\{\prod_{j=1}^{r_1} p_1 f_1(x_{1j})\right\}\left\{\prod_{j=1}^{r_2} p_2 f_2(x_{2j})\right\}\left\{\prod_{j=1}^{r_3} (1-p_1-p_2) f_3(x_{3j})\right\}\{1-F(t)\}^{n-r}$$

(5)

After simplification the likelihood function becomes:

$$L(\Psi \mid \mathrm{x}) \propto \theta_1^{2r_1+m}\theta_2^{2r_2+l}\theta_3^{2r_3+q}\sum_{k=0}^{n-r}\sum_{m=0}^{n-r-k}\sum_{s=0}^{k}\sum_{l=0}^{k-s}\sum_{q=0}^{s}\binom{n-r}{k}\binom{n-r-k}{m}\binom{k}{s}\binom{k-s}{l}\binom{s}{q} p_1^{r_1+n-r-k+m}\, p_2^{r_2+k-s+l}(1-p_1-p_2)^{r_3+s+q}$$
$$\exp(-\theta_1 2\sum_{j=1}^{r_1} x_{1j} + 2t(n-r-k) + 2tm)\exp(-\theta_2 2\sum_{j=1}^{r_2} x_{2j} + 2t(k-s) + 2tl)\exp(-\theta_3 2\sum_{j=1}^{r_3} x_{3j} + 2ts + 2tq)$$

(6)

Where $\Psi = (\theta_d, p_d),\ d = 1,2,3$ p_1, p_2 and p_3 are mixing proportions and $p_1, p_2 \ge 0$, and $X = (x_{11}, ..., x_{1r_1}, x_{21}, ..., x_{2r_2}, x_{31}, ..., x_{3r_3})$ are the observed failure times.

2.2. Posterior distributions using Jeffreys' and Gamma priors

Non-informative prior is used when no or limited information is available about the parameter. Most commonly used non-informative prior is Jeffreys' prior. Informative prior (IP) on the other hand conveys definite information about the parameter of interest. In this study gamma prior is used as an informative prior.

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2.2.1. Posterior distribution using NIP

Jeffreys' prior is considered for mixture model parameters θ_d. Jeffreys prior utilizes the Fisher information criteria and is defined as $p(\theta_d) \propto |I(\theta_d)|^{\frac{1}{2}}$, $d = 1,2,3,$ where $\mathrm{I}(\theta_d) = -E\left[\frac{\partial^2 f(x|\theta_d)}{\partial \theta_d^2}\right]$ is the Fisher's information matrix. While prior for mixing proportion p_d is assumed to be uniform over the interval $(0,1)$. The joint prior distribution of parameters θ_d and p_d (Sinha, 1998) is written as:

$$\pi_{NIP}(\Psi) \propto \prod_{d=1}^{3} \frac{1}{\theta_d}, \qquad \theta_d > 0,\ 0 \le p_d \le 1 \tag{7}$$

Using JP (Eq. (7)) and likelihood function (Eq. (6)), the joint posterior distribution becomes:

$$\rho_{post}(\Psi \mid \mathrm{x}) = D_1^{-1} \sum_{k=0}^{n-r} \sum_{m=0}^{n-r-k} \sum_{s=0}^{k} \sum_{l=0}^{k-s} \sum_{q=0}^{s} \binom{n-r}{k}\binom{n-r-k}{m}\binom{k}{s}\binom{k-s}{l}\binom{s}{q} p_1^{L_{01}-1} p_2^{M_{01}-1}(1-p_1-p_2)^{N_{01}-1} \\ \theta_1^{E_{11}-1}\theta_2^{E_{21}-1}\theta_3^{E_{31}-1} \exp(-\theta_1 H_{11})\exp(-\theta_2 H_{21})\exp(-\theta_3 H_{31}) \tag{8}$$

where

$E_{11} = 2r_1 + m, E_{21} = 2r_2 + l, E_{31} = 2r_3 + q, L_{01} = r_1 + n - r - k + m + 1, M_{01} = r_2 + k - s + l + 1,$

$N_{01} = r_3 + s + q + 1, H_{11} = 2\sum_{j=1}^{r_1} x_{1j} + 2t(n-r-k) + 2tm, H_{21} = 2\sum_{j=1}^{r_2} x_{2j} + 2t(k-s) + 2tl, H_{31} = 2\sum_{j=1}^{r_3} x_{3j} + 2ts + 2tq,$

and

$$D_1 = \Gamma(E_{11})\Gamma(E_{21})\Gamma(E_{31}) \sum_{k=0}^{n-r} \sum_{m=0}^{n-r-k} \sum_{s=0}^{k} \sum_{l=0}^{k-s} \sum_{q=0}^{s} \binom{n-r}{k}\binom{n-r-k}{m}\binom{k}{s}\binom{k-s}{l}\binom{s}{q} B(L_{01}, M_{01}, N_{01}) H_{11}^{-E_{11}} H_{21}^{-E_{21}} H_{31}^{-E_{31}}$$

2.2.2. Posterior distribution using IP

Suppose $\theta_d \sim gamma(\mathrm{v}_d, \mathrm{u}_d)$ and proportion parameters $p_d \sim BivBeta(\mathrm{u}, \mathrm{v}, \mathrm{w})$. The joint prior distribution is written as:

$$\pi_{IP}(\Psi) = \frac{v_1^{u_1}\theta_1^{u_1-1}}{\Gamma(u_1)} \exp(-v_1\theta_1) \frac{v_2^{u_2}\theta_2^{u_2-1}}{\Gamma(u_2)} \exp(-v_2\theta_2) \frac{v_3^{u_3}}{\Gamma(u_3)} \theta_3^{u_3-1} \exp(-v_3\theta_3) \frac{p_1^{u-1} p_2^{v-1}(1-p_1-p_2)^{w-1}}{B(u,v,w)} \tag{9}$$

Resulting posterior distribution using IP (Eq. (9)) and likelihood function (Eq. (6)), is given as:

$$\rho_{post}(\Psi\,|\,\mathrm{x}) = D_2^{-1}\sum_{k=0}^{n-r}\sum_{m=0}^{n-r-k}\sum_{s=0}^{k}\sum_{l=0}^{k-s}\sum_{q=0}^{s}\binom{n-r}{k}\binom{n-r-k}{m}\binom{k}{s}\binom{k-s}{l}\binom{s}{q} p_1^{L_{02}-1}p_2^{M_{02}-1}(1-p_1-p_2)^{N_{02}-1}\ \theta_1^{E_{12}-1} \theta_2^{E_{22}-1}\theta_3^{E_{32}-1}\exp(-\theta_1 H_{12})\exp(-\theta_2 H_{22})\exp(-\theta_3 H_{32}) \quad (10)$$

$$E_{12} = 2r_1 + m + u_1, E_{22} = 2r_2 + l + u_2, E_{32} = 2r_3 + q + u_3, L_{02} = r_1 + n - r - k + m + u, M_{02} = r_2 + k - s + l + v$$

$$N_{02} = r_3 + s + q + w, H_{12} = 2\sum_{j=1}^{r_1} x_{1j} + 2t(n-r-k) + 2tm + v_1$$

$$H_{22} = 2\sum_{j=1}^{r_2} x_{2j} + 2t(k-s) + 2tl + v_2, H_{32} = 2\sum_{j=1}^{r_3} x_{3j} + 2ts + 2tq + v_3,$$

$$D_2 = \Gamma(E_{12})\Gamma(E_{22})\Gamma(E_{32})\sum_{k=0}^{n-r}\sum_{m=0}^{n-r-k}\sum_{s=0}^{k}\sum_{l=0}^{k-s}\sum_{q=0}^{s}\binom{n-r}{k}\binom{n-r-k}{m}\binom{k}{s}\binom{k-s}{l}\binom{s}{q} B(L_{02}, M_{02}, N_{02})H_{12}^{-E_{12}}H_{22}^{-E_{22}}H_{32}^{-E_{32}}$$

2.3. Bayesian estimation under different loss functions

Loss function is essential component of Bayesian estimation and different loss functions are used to serve the purpose as there is no proper analytical method and rule that identifies which loss function is appropriate to be used. The symmetric loss functions are not suitable in many statistical problems, especially when we want to estimate the reliability and failure rates because overestimation will generate more loss than the underestimation. So, we use both symmetric and non-symmetric loss functions to evaluate Bayes estimators. Squared error loss function (SELF) is the most famous among symmetric loss functions in Bayesian statistics. But precautionary loss function (PLF) and DeGroot loss function (DLF) are frequently used non-symmetric loss functions. The Bayes estimator $\hat{\omega}$ is obtained by minimizing Bayes risk is defined as $\rho(\hat{\omega}) = E_{\theta|X}\{L(\theta,\hat{\omega})\}$, where $L(\theta,\hat{\omega})$ is the loss incurred in estimating θ by $\hat{\omega}$. The expressions for Bayes estimators and their risks under these loss functions are given in Table 1.

Table1. Bayes estimators and their risks under loss functions

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Loss Functions	Bayes Estimators	Bayes Risks
SELF = $L(\theta,\hat{\omega})=(\theta-\hat{\omega})^2$	$\hat{\omega}=E_{\theta\|x}(\theta)$	$\rho(\hat{\omega})=E_{\theta\|x}(\theta^2)-\{E_{\theta\|x}(\theta)\}^2$
PLF = $L(\theta,\hat{\omega})=\frac{(\theta-\hat{\omega})^2}{\hat{\omega}}$	$\hat{\omega}=\{E_{\theta\|x}(\theta^2)\}^{\frac{1}{2}}$	$\rho(\hat{\omega})=2\{E_{\theta\|x}(\theta^2)\}^{\frac{1}{2}}-2\{E_{\theta\|x}(\theta)\}$
DLF = $L(\theta,\hat{\omega})=\left(\frac{\theta-\hat{\omega}}{\hat{\omega}}\right)^2$	$\hat{\omega}=\frac{E_{\theta\|x}(\theta^2)}{E_{\theta\|x}(\theta)}$	$\rho(\hat{\omega})=1-\frac{\{E_{\theta\|x}(\theta)\}^2}{E_{\theta\|x}(\theta^2)}$

2.3.1 Bayes Estimators and Bayes Risks under SELF

In perspective of least square theory, SELF was introduced by Legendre (1806). Following Bayes estimators and Bayes risks are obtained using JP and IP under SELF:

$$\hat{\theta}_{1sv}=\frac{\Gamma(E_{1v}+1)\Gamma(E_{2v})\Gamma(E_{3v})}{D_v}\sum_{k=0}^{n-r}\sum_{m=0}^{n-r-k}\sum_{s=0}^{k}\sum_{l=0}^{k-s}\sum_{q=0}^{s}\binom{n-r}{k}\binom{n-r-k}{m}\binom{k}{s}\binom{k-s}{l}\binom{s}{q}\times H_{1v}^{-(E_{1v}+1)}H_{2v}^{-E_{2v}}H_{3v}^{-E_{3v}} B(L_{0v},M_{0v},N_{0v}) \quad (11)$$

$$\hat{\theta}_{2sv}=\frac{\Gamma(E_{1v})\Gamma(E_{2v}+1)\Gamma(E_{3v})}{D_v}\sum_{k=0}^{n-r}\sum_{m=0}^{n-r-k}\sum_{s=0}^{k}\sum_{l=0}^{k-s}\sum_{q=0}^{s}\binom{n-r}{k}\binom{n-r-k}{m}\binom{k}{s}\binom{k-s}{l}\binom{s}{q}\times H_{1v}^{-E_{1v}}H_{2v}^{-(E_{2v}+1)}H_{3v}^{-E_{3v}}B(L_{0v},M_{0v},N_{0v}) \quad (12)$$

$$\hat{\theta}_{3sv}=\frac{\Gamma(E_{1v})\Gamma(E_{2v})\Gamma(E_{3v}+1)}{D_v}\sum_{k=0}^{n-r}\sum_{m=0}^{n-r-k}\sum_{s=0}^{k}\sum_{l=0}^{k-s}\sum_{q=0}^{s}\binom{n-r}{k}\binom{n-r-k}{m}\binom{k}{s}\binom{k-s}{l}\binom{s}{q} H_{1v}^{-E_{1v}}H_{2v}^{-E_{2v}}H_{3v}^{-(E_{3v}+1)}B(L_{0v},M_{0v},N_{0v}) \quad (13)$$

$$\hat{p}_{1sv}=\frac{\Gamma(E_{1v})\Gamma(E_{2v})\Gamma(E_{3v})}{D_v}\sum_{k=0}^{n-r}\sum_{m=0}^{n-r-k}\sum_{s=0}^{k}\sum_{l=0}^{k-s}\sum_{q=0}^{s}\binom{n-r}{k}\binom{n-r-k}{m}\binom{k}{s}\binom{k-s}{l}\binom{s}{q} H_{1v}^{-E_{1v}}H_{2v}^{-E_{2v}}H_{3v}^{-E_{3v}}B(L_{0v}+1,M_{0v},N_{0v}) \quad (14)$$

$$\hat{p}_{2sv}=\frac{\Gamma(E_{1v})\Gamma(E_{2v})\Gamma(E_{3v})}{D_v}\sum_{k=0}^{n-r}\sum_{m=0}^{n-r-k}\sum_{s=0}^{k}\sum_{l=0}^{k-s}\sum_{q=0}^{s}\binom{n-r}{k}\binom{n-r-k}{m}\binom{k}{s}\binom{k-s}{l}\binom{s}{q} H_{1v}^{-E_{1v}}H_{2v}^{-E_{2v}}H_{3v}^{-E_{3v}}B(L_{0v},M_{0v}+1,N_{0v}) \tag{15}$$

And respective risks are:

$$\rho(\hat{\theta}_{1sv})=\frac{\Gamma(E_{1v}+2)\Gamma(E_{2v})\Gamma(E_{3v})}{D_v}\sum_{k=0}^{n-r}\sum_{m=0}^{n-r-k}\sum_{s=0}^{k}\sum_{l=0}^{k-s}\sum_{q=0}^{s}\binom{n-r}{k}\binom{n-r-k}{m}\binom{k}{s}\binom{k-s}{l}\binom{s}{q} H_{1v}^{-(E_{1v}+2)}H_{2v}^{-E_{2v}}H_{3v}^{-E_{3v}}B(L_{0v},M_{0v},N_{0v})-\left\{\hat{\theta}_{1sv}\right\}^2 \tag{16}$$

$$\rho(\hat{\theta}_{2sv})=\frac{\Gamma(E_{1v})\Gamma(E_{2v}+2)\Gamma(E_{3v})}{D_v}\sum_{k=0}^{n-r}\sum_{m=0}^{n-r-k}\sum_{s=0}^{k}\sum_{l=0}^{k-s}\sum_{q=0}^{s}\binom{n-r}{k}\binom{n-r-k}{m}\binom{k}{s}\binom{k-s}{l}\binom{s}{q} H_{1v}^{-E_{1v}}H_{2v}^{-(E_{2v}+2)}H_{3v}^{-E_{3v}} B(L_{0v},M_{0v},N_{0v})-\left\{\hat{\theta}_{2sv}\right\}^2 \tag{17}$$

$$\rho(\hat{\theta}_{3sv})=\frac{\Gamma(E_{1v})\Gamma(E_{2v})\Gamma(E_{3v}+2)}{D_v}\sum_{k=0}^{n-r}\sum_{m=0}^{n-r-k}\sum_{s=0}^{k}\sum_{l=0}^{k-s}\sum_{q=0}^{s}\binom{n-r}{k}\binom{n-r-k}{m}\binom{k}{s}\binom{k-s}{l}\binom{s}{q} H_{1v}^{-E_{1v}}H_{2v}^{-E_{2v}}H_{3v}^{-(E_{3v}+2)} B(L_{0v},M_{0v},N_{0v})-\left\{\hat{\theta}_{3sv}\right\}^2 \tag{18}$$

$$\rho(\hat{p}_{1sv})=\frac{\Gamma(E_{1v})\Gamma(E_{2v})\Gamma(E_{3v})}{D_v}\sum_{k=0}^{n-r}\sum_{m=0}^{n-r-k}\sum_{s=0}^{k}\sum_{l=0}^{k-s}\sum_{q=0}^{s}\binom{n-r}{k}\binom{n-r-k}{m}\binom{k}{s}\binom{k-s}{l}\binom{s}{q} H_{1v}^{-E_{1v}}H_{2v}^{-E_{2v}}H_{3v}^{-E_{3v}} B(L_{0v}+2,M_{0v},N_{0v})-\left\{\hat{p}_{1sv}\right\}^2 \tag{19}$$

$$\rho(\hat{p}_{2sv})=\frac{\Gamma(E_{1v})\Gamma(E_{2v})\Gamma(E_{3v})}{D_v}\sum_{k=0}^{n-r}\sum_{m=0}^{n-r-k}\sum_{s=0}^{k}\sum_{l=0}^{k-s}\sum_{q=0}^{s}\binom{n-r}{k}\binom{n-r-k}{m}\binom{k}{s}\binom{k-s}{l}\binom{s}{q} H_{1v}^{-E_{1v}}H_{2v}^{-E_{2v}}H_{3v}^{-E_{3v}}B(L_{0v},M_{0v}+2,N_{0v})-\left\{\hat{p}_{2sv}\right\}^2 \tag{20}$$

Where v = 1 for JP and v = 2 for GP and s represent the SELF

2.3.2 Bayes Estimators and Bayes Risks under PLF

Norstrom (1996) introduced PLF and a special case of general class of PLFs. The Bayes estimators and Bayes risks using JP and IP under PLF are given as:

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$$\hat{\theta}_{1pv} = \left\{ \begin{array}{l} \dfrac{\Gamma(E_{1v}+2)\Gamma(E_{2v})\Gamma(E_{3v})}{D_v} \displaystyle\sum_{k=0}^{n-r} \sum_{m=0}^{n-r-k} \sum_{s=0}^{k} \sum_{l=0}^{k-s} \sum_{q=0}^{s} \binom{n-r}{k} \binom{n-r-k}{m} \binom{k}{s} \binom{k-s}{l} \binom{s}{q} \times \\ H_{1v}^{-(E_{1v}+2)} H_{2v}^{-E_{2v}} H_{3v}^{-E_{3v}} B(L_{0v}, M_{0v}, N_{0v}) \end{array} \right\}^{\frac{1}{2}} \quad (21)$$

$$\hat{\theta}_{2pv} = \left\{ \begin{array}{l} \dfrac{\Gamma(E_{1v})\Gamma(E_{2v}+2)\Gamma(E_{3v})}{D_v} \displaystyle\sum_{k=0}^{n-r} \sum_{m=0}^{n-r-k} \sum_{s=0}^{k} \sum_{l=0}^{k-s} \sum_{q=0}^{s} \binom{n-r}{k} \binom{n-r-k}{m} \binom{k}{s} \binom{k-s}{l} \binom{s}{q} \times \\ H_{1v}^{-E_{1v}} H_{2v}^{-(E_{2v}+2)} H_{3v}^{-E_{3v}} B(L_{0v}, M_{0v}, N_{0v}) \end{array} \right\}^{\frac{1}{2}} \quad (22)$$

$$\hat{\theta}_{3pv} = \left\{ \begin{array}{l} \dfrac{\Gamma(E_{1v})\Gamma(E_{2v})\Gamma(E_{3v}+2)}{D_v} \displaystyle\sum_{k=0}^{n-r} \sum_{m=0}^{n-r-k} \sum_{s=0}^{k} \sum_{l=0}^{k-s} \sum_{q=0}^{s} \binom{n-r}{k} \binom{n-r-k}{m} \binom{k}{s} \binom{k-s}{l} \binom{s}{q} \\ H_{1v}^{-E_{1v}} H_{2v}^{-E_{2v}} H_{3v}^{-(E_{3v}+2)} B(L_{0v}, M_{0v}, N_{0v}) \end{array} \right\}^{\frac{1}{2}} \quad (23)$$

$$\hat{p}_{1pv} = \left\{ \begin{array}{l} \dfrac{\Gamma(E_{1v})\Gamma(E_{2v})\Gamma(E_{3v})}{D_v} \displaystyle\sum_{k=0}^{n-r} \sum_{m=0}^{n-r-k} \sum_{s=0}^{k} \sum_{l=0}^{k-s} \sum_{q=0}^{s} \binom{n-r}{k} \binom{n-r-k}{m} \binom{k}{s} \binom{k-s}{l} \binom{s}{q} \\ H_{1v}^{-E_{1v}} H_{2v}^{-E_{2v}} H_{3v}^{-E_{3v}} B(L_{0v}+2, M_{0v}, N_{0v}) \end{array} \right\}^{\frac{1}{2}} \quad (24)$$

$$\hat{p}_{2pv} = \left\{ \begin{array}{l} \dfrac{\Gamma(E_{1v})\Gamma(E_{2v})\Gamma(E_{3v})}{D_v} \displaystyle\sum_{k=0}^{n-r} \sum_{m=0}^{n-r-k} \sum_{s=0}^{k} \sum_{l=0}^{k-s} \sum_{q=0}^{s} \binom{n-r}{k} \binom{n-r-k}{m} \binom{k}{s} \binom{k-s}{l} \binom{s}{q} \\ H_{1v}^{-E_{1v}} H_{2v}^{-E_{2v}} H_{3v}^{-E_{3v}} B(L_{0v}, M_{0v}+2, N_{0v}) \end{array} \right\}^{\frac{1}{2}} \quad (25)$$

Relevant Bayes risks are:

$$\rho(\hat{\theta}_{1pv}) = 2\left\{ \begin{array}{l} \dfrac{\Gamma(E_{1v}+2)\Gamma(E_{2v})\Gamma(E_{3v})}{D_v} \displaystyle\sum_{k=0}^{n-r} \sum_{m=0}^{n-r-k} \sum_{s=0}^{k} \sum_{l=0}^{k-s} \sum_{q=0}^{s} \binom{n-r}{k} \binom{n-r-k}{m} \binom{k}{s} \binom{k-s}{l} \binom{s}{q} \\ H_{1v}^{-(E_{1v}+2)} H_{2v}^{-E_{2v}} H_{3v}^{-E_{3v}} B(L_{0v}, M_{0v}, N_{0v}) \end{array} \right\}^{\frac{1}{2}} - 2\left\{\hat{\theta}_{1sv}\right\}$$

(26)

$$\rho(\hat{\theta}_{2pv}) = 2\left\{ \begin{array}{l} \dfrac{\Gamma(E_{1v})\Gamma(E_{2v}+2)\Gamma(E_{3v})}{D_v} \displaystyle\sum_{k=0}^{n-r} \sum_{m=0}^{n-r-k} \sum_{s=0}^{k} \sum_{l=0}^{k-s} \sum_{q=0}^{s} \binom{n-r}{k} \binom{n-r-k}{m} \binom{k}{s} \binom{k-s}{l} \binom{s}{q} \\ H_{1v}^{-E_{1v}} H_{2v}^{-(E_{2v}+2)} H_{3v}^{-E_{3v}} B(L_{0v}, M_{0v}, N_{0v}) \end{array} \right\}^{\frac{1}{2}} - 2\left\{\hat{\theta}_{2sv}\right\} \quad (27)$$

$$\rho(\hat{\theta}_{3pv}) = 2\left\{\begin{matrix}\frac{\Gamma(E_{1v})\Gamma(E_{2v})\Gamma(E_{3v}+2)}{D_v}\sum_{k=0}^{n-r}\sum_{m=0}^{n-r-k}\sum_{s=0}^{k}\sum_{l=0}^{k-s}\sum_{q=0}^{s}\binom{n-r}{k}\binom{n-r-k}{m}\binom{k}{s}\binom{k-s}{l}\binom{s}{q} \\ H_{1v}^{-E_{1v}}H_{2v}^{-E_{2v}}H_{3v}^{-(E_{3v}+2)}B(L_{0v},M_{0v},N_{0v})\end{matrix}\right\}^{\frac{1}{2}} - 2\left\{\hat{\theta}_{3sv}\right\} \quad (28)$$

$$\rho(\hat{p}_{1pv}) = 2\left\{\begin{matrix}\frac{\Gamma(E_{1v})\Gamma(E_{2v})\Gamma(E_{3v})}{D_v}\sum_{k=0}^{n-r}\sum_{m=0}^{n-r-k}\sum_{s=0}^{k}\sum_{l=0}^{k-s}\sum_{q=0}^{s}\binom{n-r}{k}\binom{n-r-k}{m}\binom{k}{s}\binom{k-s}{l}\binom{s}{q} \\ H_{1v}^{-E_{1v}}H_{2v}^{-E_{2v}}H_{3v}^{-E_{3v}}B(L_{0v}+2,M_{0v},N_{0v})\end{matrix}\right\}^{\frac{1}{2}} - 2\left\{\hat{p}_{1sv}\right\} \quad (\quad 2$$

$$\rho(\hat{p}_{2pv}) = 2\left\{\begin{matrix}\frac{\Gamma(E_{1v})\Gamma(E_{2v})\Gamma(E_{3v})}{D_v}\sum_{k=0}^{n-r}\sum_{m=0}^{n-r-k}\sum_{s=0}^{k}\sum_{l=0}^{k-s}\sum_{q=0}^{s}\binom{n-r}{k}\binom{n-r-k}{m}\binom{k}{s}\binom{k-s}{l}\binom{s}{q} \\ H_{1v}^{-E_{1v}}H_{2v}^{-E_{2v}}H_{3v}^{-E_{3v}}B(L_{0v},M_{0v}+2,N_{0v})\end{matrix}\right\}^{\frac{1}{2}} - 2\left\{\hat{p}_{2sv}\right\} \quad (30)$$

Where v = 1 for JP and v = 2 for GP and p represent the PLF

2.3.3 Bayes Estimators and Bayes Risks under DLF

DeGroot loss function is attributed to DeGroot (2005). The Bayes estimators and Bayes risks using JP and IP under DLF are given as:

$$\hat{\theta}_{1dv} = \left[\begin{matrix}\Gamma(E_{1v}+2)\Gamma(E_{2v})\Gamma(E_{3v})\sum_{k=0}^{n-r}\sum_{m=0}^{n-r-k}\sum_{s=0}^{k}\sum_{l=0}^{k-s}\sum_{q=0}^{s}\binom{n-r}{k}\binom{n-r-k}{m}\binom{k}{s}\binom{k-s}{l}\binom{s}{q}B(L_{0v},M_{0v},N_{0v}) \\ H_{1v}^{-(E_{1v}+2)}H_{2v}^{-E_{2v}}H_{3v}^{-E_{3v}} / \hat{\theta}_{1sv}\end{matrix}\right]$$

(31)

$$\hat{\theta}_{2dv} = \left[\begin{matrix}\Gamma(E_{1v})\Gamma(E_{2v}+2)\Gamma(E_{3v})\sum_{k=0}^{n-r}\sum_{m=0}^{n-r-k}\sum_{s=0}^{k}\sum_{l=0}^{k-s}\sum_{q=0}^{s}\binom{n-r}{k}\binom{n-r-k}{m}\binom{k}{s}\binom{k-s}{l}\binom{s}{q}B(L_{0v},M_{0v},N_{0v}) \\ H_{1v}^{-E_{1v}}H_{2v}^{-(E_{2v}+2)}H_{3v}^{-E_{3v}} / \hat{\theta}_{2sv}\end{matrix}\right] \quad (32)$$

$$\hat{\theta}_{3dv} = \left[\begin{matrix}\Gamma(E_{1v})\Gamma(E_{2v})\Gamma(E_{3v}+2)\sum_{k=0}^{n-r}\sum_{m=0}^{n-r-k}\sum_{s=0}^{k}\sum_{l=0}^{k-s}\sum_{q=0}^{s}\binom{n-r}{k}\binom{n-r-k}{m}\binom{k}{s}\binom{k-s}{l}\binom{s}{q}B(L_{0v},M_{0v},N_{0v}) \\ H_{1v}^{-E_{1v}}H_{2v}^{-E_{2v}}H_{3v}^{-(E_{3v}+2)} / \hat{\theta}_{3sv}\end{matrix}\right] \quad (33)$$

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$$\hat{p}_{1dv} = \left[\begin{array}{l} \Gamma(E_{1v})\Gamma(E_{2v})\Gamma(E_{3v}) \sum_{k=0}^{n-r}\sum_{m=0}^{n-r-k}\sum_{s=0}^{k}\sum_{l=0}^{k-s}\sum_{q=0}^{s} \binom{n-r}{k}\binom{n-r-k}{m}\binom{k}{s}\binom{k-s}{l}\binom{s}{q} B(L_{0v}+2, M_{0v}, N_{0v}) \\ H_{1v}^{-E_{1v}} H_{2v}^{-E_{2v}} H_{3v}^{-E_{3v}} \,/\, \hat{p}_{1sv} \end{array} \right] \quad (34)$$

$$\hat{p}_{2dv} = \left[\begin{array}{l} \Gamma(E_{1v})\Gamma(E_{2v})\Gamma(E_{3v}) \sum_{k=0}^{n-r}\sum_{m=0}^{n-r-k}\sum_{s=0}^{k}\sum_{l=0}^{k-s}\sum_{q=0}^{s} \binom{n-r}{k}\binom{n-r-k}{m}\binom{k}{s}\binom{k-s}{l}\binom{s}{q} B(L_{0v}, M_{0v}+2, N_{0v}) \\ H_{1v}^{-E_{1v}} H_{2v}^{-E_{2v}} H_{3v}^{-E_{3v}} \,/\, \hat{p}_{2sv} \end{array} \right] \quad (35)$$

Bayes risks are:

$$\rho(\hat{\theta}_{1dv}) = 1 - \left[\left\{\hat{\theta}_{1sv}\right\}^2 / \left\{ \begin{array}{l} D_v \Gamma(E_{1v}+2)\Gamma(E_{2v})\Gamma(E_{3v}) \sum_{k=0}^{n-r}\sum_{m=0}^{n-r-k}\sum_{s=0}^{k}\sum_{l=0}^{k-s}\sum_{q=0}^{s} \binom{n-r}{k}\binom{n-r-k}{m}\binom{k}{s}\binom{k-s}{l}\binom{s}{q} B(L_{0v}, M_{0v}, N_{0v}) \\ H_{1v}^{-(E_{1v}+2)} H_{2v}^{-E_{2v}} H_{3v}^{-E_{3v}} \end{array} \right\} \right]$$

(36)

$$\rho(\hat{\theta}_{2dv}) = 1 - \left[\left\{\hat{\theta}_{2sv}\right\}^2 / \left\{ \begin{array}{l} D_v \Gamma(E_{1v})\Gamma(E_{2v}+2)\Gamma(E_{3v}) \sum_{k=0}^{n-r}\sum_{m=0}^{n-r-k}\sum_{s=0}^{k}\sum_{l=0}^{k-s}\sum_{q=0}^{s} \binom{n-r}{k}\binom{n-r-k}{m}\binom{k}{s}\binom{k-s}{l}\binom{s}{q} B(L_{0v}, M_{0v}, N_{0v}) \\ H_{1v}^{-E_{1v}} H_{2v}^{-(E_{2v}+2)} H_{3v}^{-E_{3v}} \end{array} \right\} \right]$$

(37)

$$\rho(\hat{\theta}_{3dv}) = 1 - \left[\left\{\hat{\theta}_{3sv}\right\}^2 / \left\{ \begin{array}{l} D_v \Gamma(E_{1v})\Gamma(E_{2v})\Gamma(E_{3v}+2) \sum_{k=0}^{n-r}\sum_{m=0}^{n-r-k}\sum_{s=0}^{k}\sum_{l=0}^{k-s}\sum_{q=0}^{s} \binom{n-r}{k}\binom{n-r-k}{m}\binom{k}{s}\binom{k-s}{l}\binom{s}{q} B(L_{0v}, M_{0v}, N_{0v}) \\ H_{1v}^{-E_{1v}} H_{2v}^{-E_{2v}} H_{3v}^{-(E_{3v}+2)} \end{array} \right\} \right] \quad (38)$$

$$\rho(\hat{p}_{1dv}) = 1 - \left[\left\{\hat{p}_{1sv}\right\}^2 / \left\{ \begin{array}{l} D_v \Gamma(E_{1v})\Gamma(E_{2v})\Gamma(E_{3v}) \sum_{k=0}^{n-r}\sum_{m=0}^{n-r-k}\sum_{s=0}^{k}\sum_{l=0}^{k-s}\sum_{q=0}^{s} \binom{n-r}{k}\binom{n-r-k}{m}\binom{k}{s}\binom{k-s}{l}\binom{s}{q} B(L_{0v}+2, M_{0v}, N_{0v}) \\ H_{1v}^{-E_{1v}} H_{2v}^{-E_{2v}} H_{3v}^{-E_{3v}} \end{array} \right\} \right] \quad (39)$$

$$\rho(\hat{p}_{2dv}) = 1 - \left[\left\{\hat{p}_{2sv}\right\}^2 / \left\{ \begin{array}{l} D_v \Gamma(E_{1v})\Gamma(E_{2v})\Gamma(E_{3v}) \sum_{k=0}^{n-r}\sum_{m=0}^{n-r-k}\sum_{s=0}^{k}\sum_{l=0}^{k-s}\sum_{q=0}^{s} \binom{n-r}{k}\binom{n-r-k}{m}\binom{k}{s}\binom{k-s}{l}\binom{s}{q} B(L_{0v}, M_{0v}+2, N_{0v}) \\ H_{1v}^{-E_{1v}} H_{2v}^{-E_{2v}} H_{3v}^{-E_{3v}} \end{array} \right\} \right] \quad (40)$$

Where v = 1 for JP and v = 2 for GP and d represent the DLF

2.4 Elicitation of hyperparameters

Elicitation is used to transform an expert's knowledge in to a joint probability distribution about a specific quantity. In Bayesian analysis, it is used to define different values of hyperparameters in a prior distribution. Many authors discussed about elicitation such as Kadane et al. (1980), Gavasakar (1988) and Hahn (2006). Aslam (2003) proposed different techniques for the elicitation of hyperparameters which are based on prior predictive distribution (PPD). In this study, method of PPD is selected to elicit the hyperparameters. The PPD for ‘x’ (random variable) may be obtained as:

$$p(\mathrm{x}) = \int_{\Upsilon} p(\mathrm{x} \mid \Upsilon)\pi_2(\Upsilon)d\Upsilon \tag{41}$$

By substituting Eqs. (3) and (9) in Eq. (41) simplification provides the following PPD :

$$p(\mathrm{x}) = \frac{4}{(u+v+w)}\left[\frac{uu_1(u_1+1)v_1^{u_1}x}{(v_1+2x)^{u_1+2}} + \frac{vu_2(u_2+1)v_2^{u_2}x}{(v_2+2x)^{u_2+2}} + \frac{wu_3(u_3+1)v_3^{u_3}x}{(v_3+2x)^{u_3+2}}\right] \tag{42}$$

The PPD given in Eq. (42) is used to consider the 9 intervals (0, 1), (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 8) and (8, 9) with probabilities 0.57, 0.20, 0.10, 0.05, 0.02, 0.015, 0.012, 0.009 and 0.004 to elicit the 9 hyperparameters. These nine intervals are solved simultaneously in Mathematica to assign numerical values to hyperparameters $(u_1, v_1, u_2, v_2, u_3, v_3, u, v, w)$. Finally, the 9 values of hyperparameters are obtained as 3.871, 3.378, 3.310, 3.078, 2.933, 2.711, 2.238, 2.400 and 1.757.

2.5 Simulation

Simulated results for Bayes estimators of 3-component mixture of the Ailamujia distribution are obtained and properties of these estimates are studied in terms of different test termination times and sample sizes. Three different sample sizes n = 25, 50, 100 are generated from 3-component mixture of the Ailamujia distributions considering parametric values $:(\theta_1, \theta_2, \theta_3, p_1, p_2) = \{(0.30, 0.25, 0.20, 0.50, 0.20), \text{and } (0.1, 0.15, 0.1, 0.40, 0.30)\}$

Sample of different sizes ($p_1 n$, $p_2 n$, and $(1 - p_1 - p_2)n$) are randomly generated from the first, second and third component ($f_1(x;\theta_1)$, $f_2(x;\theta_2)$ and $f_3(x;\theta_3)$) densities. The effect of test termination time on the Bayes estimators is determined by using the type-I right censoring scheme. The observations greater than the pre-determined test termination time $'t'$ are discarded and treated as censored. Mathematica 12 is used to get numerical features of BEs and PRs and are given in TABLES (2-5).

3. Result's discussion

From TABLES 2-5 it is noticed that when the sample size increases, Bayes estimates of component and proportion parameters become closer to its true value for both termination times at varying sample sizes. The proportion parameter ‘ p_1 ’ is under estimated using NIP and IP under SELF and PLF but it is over estimated when DLF is used. Overall under-estimation observed for component of mixture model parameters and proportion parameters is lower for larger sample sizes. On the other hand, over estimation of component and proportion parameters is higher for smaller test termination times. It is also observed that for a fixed test termination time, the Bayes risks of the Bayes estimates decreases for larger sample sizes. But if we increase the test termination time the relevant risks decreases for all the priors, sample sizes and loss functions. However for mixing proportions (p_1 and p_2), SELF is better in performance as it provides the smaller Bayes risks. So, it is concluded that SELF is a better choice for estimating the proportion parameters (p_1 and p_2). The selection of the best prior depends on the associated posterior risks. We observed that IP has less PRs (with some exceptions) as compared to NIP under SELF, PLF and DLF. Thus, we can say that IP is more efficient prior. It is also observed that performance of DLF is better for estimating parameters of component densities than the other two loss functions used.

3.1 Application to repair times data

The data from (1^{st} Oct 2019 – 31^{st} Dec 2019) on repair time (in hours) of different components used in refrigerator is taken from Haier Service Center (Rawalpindi). To demonstrate the applicability of proposed methodology, the repair time (in hours) of 3 components is included in the data set namely, compressor, thermostat and door gasket. Ailamujia distribution is a lifetime distribution and it is mostly used to model the repair time. Data set consists of 120 observations and each component contains 40 observations. Censoring time is fixed at 73.0 $(t = 73.0)$. So, the values which are greater than or equal to 73.0 are excluded or truncated and then Ailamujia mixture model is fitted to data set.

The characteristics which are extracted from censored data to obtain the estimates (BEs & PRs) for the proposed model are: n = 120, $r_1 = 38$, $r_2 = 35$, $r_3 = 35$, $r = r_1 + r_2 + r_3 = 108$, $n - r = 12$, $\sum_{j=1}^{r_1} x_{1j} = 873$, $\sum_{j=1}^{r_2} x_{2j} = 585$ and $\sum_{j=1}^{r_3} x_{3j} = 968$.

Here $n - r = 12$ indicates 10% censoring rate. BEs and BRs using both priors under SELF, PLF and DLF are presented in TABLE 6. Results represent the estimates of 3-component mixture of Ailamujia distribution which are repair times (in hours) of refrigerators. Reciprocal of the obtained estimates can be thought of average repair times of refrigerators. From the estimate of parameter of first component of the mixture model, we conclude that average time of refrigerator repair when compressor was faulty is about 23 hours and average repair time in case of problem of thermostat is found to be about 16 hours. And from the data application, it is found that it takes more repair time when appliance is out of order due to door gasket as it took on average 50 hours to get repaired in this case. From TABLE 6, it is noticed that the results derived from real data example are depicting the same pattern observed under the simulation study (with some exceptions). From the above numerical results it is confirmed that to estimate the two mixing weights (p_1 and p_2) SELF derive the smaller PRs as compared to PLF and DLF and IP have smaller posterior risks than NIP (JP).

3.2 Application to recovery times data

Remission (recovery) times (in months) of bladder cancer patients are taken from Lee and Wang (2003). Data set consists of 128 observations where censoring time is fixed at 21.80 $(t = 21.80)$. The values which are greater than or equal to $(\geq)$ 21.80 are excluded or truncated. Ailamujia distribution is a lifetime distribution and it is used in reliability and supportability data analysis in the field of medical science. So, the Ailamujia mixture model is fitted to this data set. The observations of this data set are divided in to three groups, 'Group-I', 'Group-II' and 'Group-III' to accommodate 3-component mixture model.

Group-I	Group-II	Group-III
0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 0.20, 2.22, 3.52, 4.98, 6.99, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 0.81, 2.62, 3.82, 5.32.	7.32, 10.06, 14.77, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 1.35.	2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63.

Necessary calculations to obtain the estimates (BEs & PRs) for proposed estimators, extracted from censored data are:

$n=128,\quad r_1=39,\quad r_2=37,\quad r_3=41,\quad r=r_1+r_2+r_3=117,\quad n-r=11,\quad \sum_{j=1}^{r_1}x_{1j}=219.78,$

$\sum_{j=1}^{r_2}x_{2j}=261.91$ and $\sum_{j=1}^{r_3}x_{3j}=321.68$.

Here, $n-r=11$ and we can say that we have almost 8.59% censored sample. Bayes estimates and the posterior risks by using JP and GP under SELF, PLF and DLF are presented in TABLE 7.

Bayes estimators presented in TABLE 7 represent the estimates of 3-component mixture of Ailamujia distribution for remission (recovery) times of the bladder cancer patients. We can take the reciprocal of estimates as average remission (recovering) times of the bladder cancer patients. After, obtaining the estimate of parameter of all three components of the mixture model, we conclude that the average remission time (recovery time) of bladder cancer patients of Group-I, Group-II and Group-III is 6 months, 8 months and 12 months respectively. The average recovery time of the patients of Group-III is more than the patients of other two groups because they are taking more time (one year) to recover from illness. After observing results we can say that it was a right decision of choosing a lifetime distribution.

From TABLE 7, it is noticed that results derived are depicting the same pattern observed under the simulation study. Here, GP derive smaller posterior risks than JP. The tabulated results also confirm that DLF is better option for estimating the 3 component parameters and SELF is best option to estimate the two mixing proportions (p_1 and p_2). From the numerical results of table 7, we also find that GP have smaller posterior risks than JP and SELF have smaller posterior risks than the other two loss functions (PLF and DLF). So, in order to make predictions of future values and for estimation of different parameters, the use of GP under SELF may be preferred.

4. Conclusion

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A 3-CAMM distribution is considered and analyzed. The joint posteriors assuming Jeffreys and gamma priors are derived. To obtain the Bayes estimates and their associated Bayes risks, three different loss functions, namely, SELF, PLF and DLF have been considered. Simulated Bayes estimates are obtained and their performance is compared in terms of different loss functions, sample sizes, priors and test termination times. We conclude from simulation results that Bayes (posterior) risks decrease by increasing the sample size. Also, the simulated results show that DLF is better option for estimating the mixture parameters $(\theta_1, \theta_2, \theta_3)$ and SELF is better option for estimating the two mixing proportions (p_1 and p_2). IP is more efficient and better prior than NIP because it has smaller Bayes (posterior) risks. The numerical results from real data reveal the same trend as observed in simulated results.

Table-2: Bayes estimates (BEs) and Bayes risks (BRs) in bold using JP when $\theta_1 = 0.30,\ \theta_2 = 0.25,\ \theta_3 = 0.20,\ p_1 = 0.50,\ p_2 = 0.20$ **and** $t = 2$

n		$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
25	SELF	1.205290	1.002320	0.907820	0.455753	0.236788
		0.094184	**0.284665**	**0.135750**	**0.009613**	**0.008148**
	PLF	1.213490	1.154770	0.985284	0.469296	0.252386
		0.075111	**0.209180**	**0.134903**	**0.021174**	**0.032722**
	DLF	1.282860	1.272770	1.071190	0.479873	0.269549
		0.062336	**0.193184**	**0.147198**	**0.044456**	**0.130243**
50	SELF	1.176530	0.899155	0.829614	0.477963	0.231910
		0.045339	**0.120129**	**0.076163**	**0.005363**	**0.004876**
	PLF	1.197250	1.004590	0.877950	0.484960	0.239395
		0.038199	**0.120750**	**0.085322**	**0.011077**	**0.019971**
	DLF	1.221440	1.073210	0.927918	0.490614	0.250130
		0.032006	**0.132840**	**0.099334**	**0.022555**	**0.084467**

100	SELF	1.181680	0.874626	0.788656	0.483241	0.224495
		0.022365	**0.062059**	**0.039409**	**0.002697**	**0.002638**
	PLF	1.218940	0.914649	0.762062	0.479840	0.232023
		0.019127	**0.068530**	**0.049593**	**0.005748**	**0.012427**
	DLF	1.178450	0.949989	0.860061	0.4830	0.239252
		0.017775	**0.091540**	**0.057213**	**0.012401**	**0.051247**

Table- 3:Bayes estimates (BEs) and Bayes risks (BRs) using JP when $\theta_1 = 0.1,\ \theta_2 = 0.15,\ \theta_3 = 0.1,\ p_1 = 0.40,\ p_2 = 0.30$ **and** $t = 3$

n		$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
25	SELF	0.552868	0.680926	0.504785	0.345349	0.327352
		0.072274	0.083704	0.085590	0.014999	0.012805
	PLF	0.591298	0.722784	0.605289	0.374548	0.348625
		0.100252	0.098833	0.137364	0.039951	0.037407
	DLF	0.651897	0.767128	0.653465	0.392641	0.363511
		0.190617	0.145953	0.23980	0.109177	0.105994
50	SELF	0.539253	0.644715	0.431361	0.342512	0.307707
		0.035005	0.037645	0.042880	0.010931	0.008787

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	PLF	0.559218	0.689104	0.493867	0.368178	0.316525
		0.064703	0.059338	0.092180	0.029704	0.026202
	DLF	0.611296	0.718503	0.531801	0.375795	0.334902
		0.128517	0.102510	0.216756	0.085977	0.079914
100	SELF	0.504901	0.682260	0.407155	0.358539	0.279408
		0.019147	0.019847	0.024700	0.007079	0.004913
	PLF	0.537114	0.661219	0.417084	0.357124	0.302282
		0.044574	0.028970	0.059713	0.022750	0.011972
	DLF	0.620326	0.723707	0.388693	0.329999	0.301911
		0.071703	0.05630	0.154098	0.065863	0.045966

Table-4: Bayes estimates (BEs) and Bayes risks (BRs) using IP when $\theta_1 = 0.30,\ \theta_2 = 0.25,\ \theta_3 = 0.20,\ p_1 = 0.50,\ p_2 = 0.20$ **and** $t = 2$

n		$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
25	SELF	1.171770	0.932399	0.917750	0.448975	0.256855
		0.076599	**0.148378**	**0.106736**	**0.008577**	**0.007576**
	PLF	1.203160	1.036170	0.979489	0.460339	0.270601
		0.063649	**0.144109**	**0.110103**	**0.019017**	**0.028067**
	DLF	1.235640	1.105980	1.025180	0.468552	0.287073
		0.053738	**0.14630**	**0.118132**	**0.041434**	**0.103631**
50	SELF	1.165770	0.926850	0.827544	0.472069	0.238824
		0.041648	**0.097112**	**0.063372**	**0.005012**	**0.004412**
	PLF	1.187820	0.949360	0.877304	0.478020	0.251454
		0.035526	**0.100276**	**0.074783**	**0.010579**	**0.018297**
	DLF	1.205430	1.003220	0.940032	0.483598	0.260804
		0.030936	**0.109421**	**0.086743**	**0.022062**	**0.072750**

100	SELF	1.153570	0.946538	0.758116	0.470147	0.224905
		0.021436	**0.062052**	**0.033656**	**0.002637**	**0.002551**
	PLF	1.24070	0.891234	0.792378	0.473054	0.239953
		0.018336	**0.078688**	**0.050242**	**0.005460**	**0.012599**
	DLF	1.196310	0.928624	0.860322	0.478999	0.247478
		0.016094	**0.088253**	**0.065291**	**0.011788**	**0.051262**

Table-5: Bayes estimates (BEs) and Bayes risks (BRs) using IP when $\theta_1 = 0.1,\ \theta_2 = 0.15,\ \theta_3 = 0.1,\ p_1 = 0.40,\ p_2 = 0.30$ **and** $t = 3$

n		$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
25	SELF	0.639577	0.688616	0.547398	0.340366	0.347010
		0.059002	**0.062410**	**0.070035**	**0.011849**	**0.011574**
	PLF	0.688223	0.724293	0.616709	0.356774	0.365362
		0.097446	**0.090837**	**0.121959**	**0.036292**	**0.033418**
	DLF	0.711666	0.795075	0.698809	0.380734	0.381353
		0.150088	**0.136165**	**0.205076**	**0.099653**	**0.089383**
50	SELF	0.582550	0.671267	0.436113	0.338018	0.315835
		0.035943	**0.038305**	**0.043923**	**0.010299**	**0.008671**
	PLF	0.605031	0.691771	0.524335	0.355850	0.336995
		0.066240	**0.063633**	**0.089421**	**0.029701**	**0.026394**
	DLF	0.668096	0.711103	0.561086	0.365444	0.353913
		0.116399	**0.104615**	**0.203067**	**0.078965**	**0.072698**
100	SELF	0.658877	0.719778	0.365034	0.314393	0.297123
		0.018693	**0.018353**	**0.019425**	**0.005111**	**0.004346**
	PLF	0.587324	0.707958	0.378229	0.346142	0.285224
		0.032885	**0.027015**	**0.054461**	**0.017102**	**0.011985**

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	DLF	0.612876	0.682443	0.524453	0.362875	0.337370
		0.093776	**0.082066**	**0.140832**	**0.060965**	**0.045270**

Table-6: Bayes estimates (BEs) and Bayes risks (BRs) using JP and IP for repair times data

Prior		$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
JP	SELF	0.041684	0.059527	0.019542	0.322028	0.293045
		0.000032	**0.000053**	**0.000007**	**0.001848**	**0.001674**
	PLF	0.042068	0.059969	0.019719	0.324885	0.295887
		0.000767	**0.000883**	**0.000354**	**0.005714**	**0.005686**
	DLF	0.042455	0.060413	0.019897	0.327767	0.298757
		0.018144	**0.014666**	**0.017865**	**0.017511**	**0.019120**
IP	SELF	0.044116	0.062247	0.020207	0.322112	0.296171
		0.000031	**0.000052**	**0.000006**	**0.001771**	**0.001639**
	PLF	0.044477	0.062685	0.020371	0.324850	0.298925
		0.000722	**0.000875**	**0.000327**	**0.005476**	**0.005507**
	DLF	0.044841	0.063125	0.020535	0.327611	0.301704
		0.016164	**0.013907**	**0.015975**	**0.016785**	**0.018338**

Table-7: BEs and PRs using UP, JP and GP for real life data set at $t = 21.80$ under SELF, PLF and DLF

Prior			$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	$\hat{p}_1$	$\hat{p}_2$
JP	SELF	BE	0.174729	0.127229	0.079830	0.306660	0.302813
		PR	(0.000441)	(0.000539)	(0.000206)	(0.001624)	(0.001966)
	PLF	BE	0.175988	0.129330	0.081108	0.309296	0.306041
		PR	(0.002517)	(0.004202)	(0.002557)	(0.005272)	(0.006457)

	DLF	BE	0.177256	0.131466	0.082407	0.311955	0.309304
		PR	(0.014252)	(0.032228)	(0.031274)	(0.016972)	(0.020988)
IP	SELF	BE	0.182557	0.134196	0.081318	0.307871	0.303726
		PR	(0.000440)	(0.000534)	(0.000192)	(0.001583)	(0.001860)
	PLF	BE	0.183784	0.136170	0.082492	0.310431	0.306774
		PR	(0.002454)	(0.003947)	(0.002347)	(0.005121)	(0.006095)
	DLF	BE	0.185019	0.138172	0.083682	0.313013	0.309852
		PR	(0.013310)	(0.028775)	(0.028250)	(0.016429)	(0.019769)

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